

An extension of Furstenberg's structure theorem for Noetherian modules and multiple recurrence theorems III

Xiongping Dai

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

Abstract

Using a recent Furstenberg structure theorem, we obtain Multiple Recurrence Theorems relative to any locally compact second countable Noetherian module G over a syndetic ring R , which generalizes Furstenberg's multiple recurrence theorem. In addition we study the multiple Birkhoff center and the pointwise multiple recurrence of a topological G -action on a compact metric space.

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0. Introduction

As a subsequent work of [14, 15], this paper will continue to study the Furstenberg multiple recurrence of dynamical systems on a standard Borel probability space induced by any locally compact second countable Noetherian module over a syndetic ring.

0.1. Basic notions

First of all, by a “lcscN” R -module $(G, +)$ over a “syndetic” ring $(R, +, \cdot)$, we mean that $(G, +)$ and $(R, +)$ both are locally compact second countable Hausdorff topological groups satisfying the following conditions (cf. [14]):

- The multiplication operation of R on G , $(t, g) \mapsto tg$, is continuous from $R \times G$ to G .
- G is a *Noetherian* R -module: for every sequence $G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$ of R -submodules of G , we have $G_n = G_{n+1}$ as n sufficiently large.
- R is *syndetic*: $\forall t \neq 0$, Rt is syndetic in the sense that one can find a compact subset K of R with $K + Rt = R$.

Moreover, we shall say that

- an R -module G is *irreducible* if $tg \neq \mathbf{o}$ whenever $R \ni t \neq 0$ and $G \ni g \neq \mathbf{o}$, where \mathbf{o} is the zero element of G and 0 is the zero element of R .

Clearly, $(\mathbb{Z}^n, +)$ over \mathbb{Z} , $(\mathbb{Q}^n, +)$ over \mathbb{Q} and $(\mathbb{R}^d, +)$ over \mathbb{R} all are irreducible lcscN modules over syndetic rings. Moreover, the p -adic integer (syndetic) ring \mathbb{Z}_p and the p -adic number (syndetic) field \mathbb{Q}_p are irreducible lcscN as module over themselves. See [34] for more examples.

Email address: xpdai@nju.edu.cn (Xiongping Dai)

0.2. Measure-theoretic multiple recurrence theorems

Let $(G, +)$ be a topological R -module and let (X, \mathcal{X}, μ) be a probability space. We will consider a measure-preserving G -action dynamical system:

$$T : G \times X \rightarrow X \quad \text{or write} \quad G \curvearrowright_T X$$

where $T_g : X \rightarrow X$ is μ -preserving for each $g \in G$ and the G -action map $T : (g, x) \mapsto T_g(x)$ is jointly measurable. For our convenience, we will write

$$T_{tg}(x) = T_g^t(x) = g^t x \quad \text{and so} \quad T_{g+h}^t = g^t \circ h^t \quad \forall t \in R \text{ and } g, h \in G$$

if no confusion. Then given any $g \neq o$,

$$T_g : R \times X \rightarrow X \quad \text{by} \quad (t, x) \mapsto T_g^t(x) = g^t x$$

defines a new μ -preserving Borel R -system.

Harry Furstenberg in 1977 [22] extraordinarily extended the classical Poincaré recurrence theorem to the *multiple recurrence* as follows:

- For any cyclic measure-preserving system (X, \mathcal{X}, μ, T) and any integer $l \geq 2$, the transformations T, T^2, \dots, T^l have a common power satisfying $\mu(A \cap T^{-n}A \cap \dots \cap T^{-ln}A) > 0$ for any set of positive μ -measure.

There he also showed that this result implies Szemerédi's theorem asserting that any set of integers of positive upper density contains arbitrarily long arithmetic progressions [39]. In 1978 [26], he and B. Weiss proved a topological analogue—the multiple Birkhoff recurrence theorem:

- If T is a homeomorphism of a compact metric space X , then for any $\varepsilon > 0$ and any integer $l = 1, 2, \dots$, there is a point $x \in X$ and a common power n of T, T^2, \dots, T^l such that $d(x, T^n x) < \varepsilon, \dots, d(x, T^{ln} x) < \varepsilon$.

This weaker result in turn implies van der Waerden's theorem on arithmetic progressions for partitions/colorings of the integers. In fact this topological result is true for any l commuting continuous transformations (cf. [23, Theorem 2.6]). In 1978 [24], he and Y. Katznelson showed that the measure-theoretic model of the multiple Birkhoff recurrence theorem is also true for arbitrary commuting transformations (cf. [24, Theorem A] and [23, Theorems 7.13, 7.14 and 7.15]). A subsequent corollary is the multidimensional extension of Szemerédi's theorem on arithmetic progressions (cf. [24, Theorem B] and [23, Theorem 7.16]).

Starting with Furstenberg's ergodic-theoretic methods introduced in his landmark papers [21, 22], there have already been many important generalizations of the multiple recurrence theorem and Szemerédi's theorem; see, e.g., [25, 5, 35, 6, 4, 19, 18, 30, 43, 40, 32, 1, 12, 38, 20, 8, 33] and references therein.

By $|\cdot|$ we mean a Haar measure on the lsc ring $(R, +, \cdot)$. Recall that a sequence of nonnull compact subsets F_n of R such that

$$\lim_{n \rightarrow \infty} \frac{|(r + F_n) \triangle F_n|}{|F_n|} = 0 \quad \forall r \in R,$$

is called a *weak Følner sequence* in $(R, +)$.¹ Particularly, a weak Følner sequence $\{F_n\}_1^\infty$ in $(R, +)$ is said to be *asymptotic* if for each $m \in \mathbb{N}$ there exists a constant $c_m > 0$ such that

$$mF_n \subseteq F_{mn} \quad \text{and} \quad |F_{mn}| \leq c_m |F_n| \quad \forall n \geq 1. \quad (AF)$$

For example, $F_n = \{0, 1, 2, \dots, n-1\}, n = 1, 2, \dots$ is an asymptotic Følner sequence in $(\mathbb{Z}, +)$; $F_n = [0, n], n = 1, 2, \dots$ is an asymptotic Følner sequence in $(\mathbb{R}, +)$ with the Euclidean topology.

In different flavor, this paper is mainly to prove the following measure-theoretic multiple recurrence theorem respecting to an irreducible lcscN module such as $\mathbb{Q}^n, \mathbb{Z}_p, \mathbb{Q}_p^n, \mathbb{R}^n$ beyond the integer group \mathbb{Z} .

Theorem 0.1. *Let G be an irreducible lcscN R -module over a syndetic ring $(R, +, \cdot)$. Then every $G \curvearrowright_T (X, \mathcal{X}, \mu)$ is an Sz-system over asymptotic Følner sequences; i.e., for any $A \in \mathcal{X}$ with $\mu(A) > 0$ and any $g_1, \dots, g_l \in G, l \geq 2$,*

$$\liminf_{n \rightarrow +\infty} \frac{1}{|F_n|} \int_{F_n} \int_X g_1^t 1_A \cdots g_l^t 1_A d\mu dt > 0,$$

over any asymptotic Følner sequence $\{F_n\}_1^\infty$ in $(R, +)$.

The \liminf is actually a limit in the classical case that $(R, +, \cdot) = (\mathbb{Z}, +, \cdot)$ and $G = \mathbb{Z}^n$; see [30, 43, 40, 1]. However, this question is open in the present context.

We can obtain from Theorem 0.1 the continuous-time version of Furstenberg's Multiple Recurrence Theorem [22, 24, 23] as follows:

Corollary 0.2. *Let $\varphi: \mathbb{R}^d \times (X, \mathcal{X}, \mu) \rightarrow (X, \mathcal{X}, \mu)$ be a Borel flow over a standard Borel probability space (X, \mathcal{X}, μ) . Then for $A \in \mathcal{X}$ with $\mu(A) > 0$,*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mu(A \cap g_1^{-t} A \cap \cdots \cap g_l^{-t} A) dt > 0$$

for any nonzero elements $g_1, \dots, g_l \in \mathbb{R}^d$ and $l \geq 1$.

Our condition (AF) is similar to Calderon's original doubling condition [10] that is formulated for an increasing family of compact symmetric neighborhoods of the zero 0 of R . We note that this condition is often incompatible with the weak Følner property, as noted in [29]. In view of this reason, we now restate our theorem as follows:

Theorem 0.3. *Let G be any lcscN R -module over a syndetic ring $(R, +, \cdot)$. Then every measure-preserving G -system $G \curvearrowright_T (X, \mathcal{X}, \mu)$ satisfies the following multiple recurrence property:*

- *For any $A \in \mathcal{X}$ with $\mu(A) > 0$ and any finite set $F = \{g_1, \dots, g_l\} \subseteq G$, there exists some integer $M = M(A, F) \geq 1$ such that for any weak Følner sequence $\{F_n\}_1^\infty$ in $(R, +)$,*

$$\limsup_{n \rightarrow +\infty} \frac{1}{|F_n|} \int_{F_n} \int_X g_1^m 1_A \cdots g_l^m 1_A d\mu dt > 0,$$

for some integer m with $1 \leq m \leq M$.

¹Here and in the future, unlike the case in literature, for a weak Følner sequence we neither need to require

$$\lim_{n \rightarrow \infty} \frac{|(K + F_n) \triangle F_n|}{|F_n|} = 0 \quad \forall K \subset R \text{ compact}$$

nor other regularities such as Tempelman condition [41] or Shulman condition (cf. [16]).

For simplicity, we shall call the property described in Theorem 0.3 the *weak Sz-property*. Following Furstenberg's correspondence principle, this weak Sz-property is already enough to derive Szemerédi's theorem.

Outline of the proof of Theorems 0.1 and 0.3 First we note that because $\{1, r_1 t, \dots, r_l t\}$ as a family of polynomials, for distinct $r_1, \dots, r_l \in \mathbb{R}$, is not \mathbb{R} -independent for $l \geq 2$, even the special case of $d = 1$ in Corollary 0.2 above is not included in [38, Theorem 1.2] by using nilflows.

On the other hand, since an irreducible lcscN ring $(R, +, \cdot)$ does not need to be the classic field \mathbb{R} and so there is no the discrete-time expression: given any $\delta \neq 0$, for any $t \in R$ we have $t = n_t \delta + r_t$ with $n_t \in \mathbb{Z}$ and $0 \leq r_t < \delta$, hence the discretization methods developed recently by Bergelson et al. [7] does not work for Theorem 0.1 here.

To prove our Multiple Recurrence Theorems in the probabilistic settings (Theorems 0.1 and 0.3), our main tool is the following structure theorem.

Furstenberg Structure Theorem ([14]). *Let G be an lcscN R -module over a syndetic ring R and $X = (X, \mathcal{X}, \mu, G)$ a nontrivial standard Borel G -system. Then there exists an ordinal η and a system of factors $\{\pi_\xi: X \rightarrow X_\xi\}_{\xi \leq \eta}$ such that*

- (a) X_0 is the one-point G -system and $X_\eta = X \pmod{\mu}$.
- (b) If $0 \leq \theta < \xi \leq \eta$, then there is a factor G -map $\pi_{\xi, \theta}: X_\xi \rightarrow X_\theta$ with $\pi_\theta = \pi_{\xi, \theta} \circ \pi_\xi$.
- (c) For each ordinal ξ with $0 \leq \xi < \eta$, $\pi_{\xi+1, \xi}: X_{\xi+1} \rightarrow X_\xi$ is a nontrivial “primitive” extension.
- (d) If ξ is a limit ordinal $\leq \eta$, then $X_\xi = \varprojlim_{\theta < \xi} X_\theta$.

Moreover, the intermediate factors are of the form

$$X_\xi = (X, \mathcal{X}_\xi, \mu, G), \quad \pi_\xi = Id_X \quad \text{and} \quad \pi_{\xi+1, \xi} = Id_{X_\xi} \quad (0 < \xi < \eta).$$

Here we refer to

$$X \rightarrow X_\eta \rightarrow \dots \rightarrow X_{\xi+1} \rightarrow X_\xi \rightarrow \dots \rightarrow X_1 \rightarrow X_0$$

a “Furstenberg factors chain” of X .

Since X_0 is an Sz-system (and X_1 is a Kh-system [15]), then as in the classical case, based on this Structure Theorem we mainly need to build three ladders to lift the Sz-property of X_1 :

- (1). For any limit factor X_ξ as in (d), if each $X_\theta, \theta < \xi$, is an (resp. a weak) Sz-system, X_ξ is also an (resp. a weak) Sz-system (Corollary 1.4 and Lemma 1.5 in §2).
- (2). If an intermediate factor $X_{\xi+1} = (X, \mathcal{X}_{\xi+1}, \mu, G)$ of X is a totally relatively weak-mixing extension of an (resp. a weak) Sz-system X_ξ , then $X_{\xi+1}$ is an (resp. a weak) Sz-system; see Corollary 2.3 in §2.
- (3). Consider any primitive link $\pi_{\xi+1, \xi}: X_{\xi+1} \rightarrow X_\xi$ for any order $\xi \geq 1$, where X_ξ is an (resp. a weak) Sz-system. See Propositions 3.5 and 3.6 in §3.

Then we can easily prove our Multiple Recurrence Theorems by combining the above three ladders; see §4.

0.3. Topological multiple recurrence

Next we will present some simple applications of Theorems 0.1 and 0.3. Clearly Theorem 0.1 implies the following Topological Multiple Recurrence Theorem, which is an extension and strengthening of Furstenberg and Weiss [26, Theorem 1.5]:

Proposition 0.4. *Let $T: G \times X \rightarrow X$ be a topological dynamical system on a compact metric space X , where G is an irreducible lcscN module over a syndetic ring R . If $G \curvearrowright_T X$ is a weak E-system (i.e. X is just the support of some T -invariant Borel probability measure of X but (X, T) does not need to be topologically transitive), then for any $g_1, \dots, g_l \in G$ and any nonempty open subset U of X ,*

$$N_{g_1, \dots, g_l}(U) = \{t \in R \mid U \cap g_1^{-t}U \cap \dots \cap g_l^{-t}U \neq \emptyset\}$$

is of positive lower density over any asymptotic Følner sequence in $(R, +)$.

For the classical case where $G = \mathbb{Z}^d$, regarded as an lcscN \mathbb{Z} -module, acts minimally on X and only requiring $N_{g_1, \dots, g_l}(U) \cap \mathbb{N} \neq \emptyset$, see [26, 23] by using Bowen's lemma and homogeneous sets, [3] also [28, Theorem 1.56] by using Ellis enveloping semigroup theory, and [9] for topological proof of the topological multiple recurrence theorem.

Our further applications need the following notation.

Definition 0.5. A measurable subset S of an lcsc ring $(R, +, \cdot)$ is called a Følner ∞ -set in R if for any weak Følner sequence $\{F_n\}_1^\infty$ in $(R, +)$, there is some integer $m \geq 1$ such that the set $\{t \in R; mt \in S\}$ is of positive upper density over $\{F_n\}_1^\infty$.

Then Theorem 0.3 implies the following

Proposition 0.6. *Let $T: G \times X \rightarrow X$ be a topological dynamical system on a compact metric space X , where G is an lcscN module over a syndetic ring R . If $G \curvearrowright_T X$ is a weak E-system, then for any sample elements $g_1, \dots, g_l \in G$ and any nonempty open subset U of X , $N_{g_1, \dots, g_l}(U)$ is a Følner ∞ -set in R .*

Based on Def. 0.5 we next introduce a concept—multiply nonwandering motion—for any topological dynamical system $G \curvearrowright_T X$ on a compact metric space X , which is the weakest multiple recurrence and which generalizes and strengthens the classical single nonwandering motion [36].

Definition 0.7. We say that a point $p \in X$ is *multiply nonwandering* for $G \curvearrowright_T X$ if for any neighborhood U of p , any sample elements $g_1, \dots, g_l \in G$, the set of multiple return times

$$N_{g_1, \dots, g_l}(U) = \{t \in R \setminus \{0\} : U \cap g_1^{-t}U \cap \dots \cap g_l^{-t}U \neq \emptyset\},$$

is of Følner ∞ -set in R . By $\Omega_F(G \curvearrowright_T X)$ it means the set of all the multiply nonwandering points of $G \curvearrowright_T X$. Here the subscript F is for Furstenberg because of his multiple recurrence.

Requiring only $N_{g_1, \dots, g_l}(U) \neq \emptyset$ in place of our Følner ∞ -set, this notion in fact was first introduced by Balcar et al. in 1987 [2] in the more general situation of semigroup actions.

Although a multiply nonwandering point might have no multiple recurrence itself, yet multiple recurrence (even periodic motion) occurs near p . By definitions, it is easy to check the following basic fact:

Lemma 0.8. *Let G be any lcscN R -module over a syndetic ring $(R, +, \cdot)$. $\Omega_F(G \curvearrowright_T X)$ is an T -invariant closed subset of X for any topological dynamical system $G \curvearrowright_T X$.*

As a simple consequence of Theorem 0.3, we can easily obtain the following result in the topological settings by considering the density points of ergodic probability measures.

Proposition 0.9. *Let G be any lcscN R -module over a syndetic ring R . Then for any topological dynamical system $G \curvearrowright_T X$ on a compact metric space X , $\Omega_F(G \curvearrowright_T X)$ is of full probability.*

Proof. Given any ergodic measure μ of $G \curvearrowright_T X$, let $\text{supp}(\mu)$ be the support of μ . Then $\text{supp}(\mu)$ is of μ -measure 1 and for any $x \in \text{supp}(\mu)$ and any neighborhood U of x we have $\mu(U) > 0$. Thus by Theorem 0.3, it follows that $N_{g_1, \dots, g_l}(U)$ is a Følner ∞ -set in R for any finite set of sample elements $g_1, \dots, g_l \in G$. This concludes the proof of Proposition 0.9. \square

Because there always exists an ergodic Borel probability of $G \curvearrowright_T X$ for G is amenable and X is a compact metric space, $\Omega_F(G \curvearrowright_T X)$ is a nonempty subset of X .

In what follows we assume $G \curvearrowright_T X$ is a topological dynamical system on a compact metric space. Next let us note that the restriction of T to $\Omega_F(G \curvearrowright_T X)$ is itself a topological G -space and further it has its own nonempty multiply nonwandering point set $\Omega_F(G \curvearrowright_T (\Omega_F(G \curvearrowright_T X)))$. We set

$$\Omega_1(T) = \Omega_F(G \curvearrowright_T X), \Omega_2(T) = \Omega_F(G \curvearrowright_T \Omega_1(T)), \dots, \Omega_{\xi+1}(T) = \Omega_F(G \curvearrowright_T \Omega_\xi(T)), \dots$$

and if ξ is a limit ordinal, we define

$$\Omega_\xi(T) = \bigcap_{\theta < \xi} \Omega_\theta(T) \quad (\neq \emptyset, T\text{-invariant and closed}).$$

So by transfinite induction we can get a chain (possibly transfinite):

$$\Omega_1(T) \supseteq \Omega_2(T) \supseteq \dots \supseteq \Omega_n(T) \supseteq \Omega_{n+1}(T) \supseteq \dots \supseteq \Omega_\omega(T) \subseteq \Omega_{\omega+1}(T) \supseteq \dots$$

Since X is a compact Hausdorff space, then from the Cantor-Baire theorem it follows that there exists an ordinal $\gamma \geq 1$ such that

$$\Omega_\gamma(T) = \Omega_F(G \curvearrowright_T \Omega_\gamma(T)).$$

Definition 0.10. As in the classical case (cf., e.g., [36, Chap. V.5]), such $\Omega_\gamma(T)$ is referred to as the *set of multiple center motions* or simply the *multiple Birkhoff center* of order γ .

Obviously, the multiple Birkhoff center $\Omega_\gamma(T)$ is an T -invariant, compact, and full probability set by Proposition 0.9 and its order γ is not greater than a transfinite number of class II.

Using the multiple Birkhoff center $\Omega_\gamma(T)$ as our tool, we shall show that there exists a “large” set of multiply recurrent points in any topological G -space in §5.2. We note that the multiple Birkhoff center that is defined in a weaker manner of lacking the Følner ∞ -set has been studied recently by Kwietniak et al [33] using different approaches (cf. e.g. [23, §1.8 and §3.1]) for \mathbb{Z} -action topological dynamical systems.

The remainder of this paper will be organized as follows.

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1. Inverse limits of Sz-systems and weak Sz-systems

In this section, let $(G, +)$ be an lcscN R -module over a syndetic ring $(R, +, \cdot)$. Let (X, \mathcal{X}, μ) be any G -space with the G -action map T and write $X = G \curvearrowright_T (X, \mathcal{X}, \mu)$ and simply write $T_g^t = g^t$ for any $t \in R$ and $g \in G$.

First, following Hillel Furstenberg [23] we introduce the following concept.

Definition 1.1. We say that X is an *Sz-system* over all asymptotic Følner sequences in $(R, +)$, where Sz is for Szemerédi, provided that for any integer $l \geq 2$,

- whenever $A \in \mathcal{X}$ with $\mu(A) > 0$ and $g_1, \dots, g_l \in G$, then

$$\liminf_{n \rightarrow +\infty} \frac{1}{|F_n|} \int_{F_n} \mu(g_1^{-t}A \cap \dots \cap g_l^{-t}A) dt > 0 \quad (1.1)$$

over any asymptotic Følner sequence $\{F_n\}_1^\infty$ in $(R, +)$;

or equivalently,

- whenever $f \in L^\infty(X, \mathcal{X}, \mu)$ with $f \geq 0$ a.e. and $\int_X f d\mu > 0$ and $g_1, \dots, g_l \in G$, then

$$\liminf_{n \rightarrow +\infty} \frac{1}{|F_n|} \int_{F_n} \int_X g_1^t f \dots g_l^t f d\mu dt > 0 \quad (1.2)$$

over any asymptotic Følner sequence $\{F_n\}_1^\infty$ in $(R, +)$.

We note that if G is an \mathbb{Z} -module and $F_n = \{1, 2, \dots, n\}$, then this definition reduces to the classical case of Furstenberg [23, Definition 7.1].

We will need the following simple equivalence later on.²

²There is a similar characterization to the property (1.2) by an almost same argument with $\int_X g_1^t f \dots g_l^t f d\mu$ in place of $\int_X g_1^t 1_A \dots g_l^t 1_A d\mu$.

Lemma 1.2. *Let $\{F_n\}_1^\infty$ be any asymptotic Følner sequence in $(R, +)$. Then an $A \in \mathcal{X}$ with $\mu(A) > 0$ has the property (1.1) over $\{F_n\}_1^\infty$ if and only if there exist $\delta > 0, \varepsilon > 0$, and a Borel subset \mathcal{H} of R such that*

$$D_*(\mathcal{H}) := \liminf_{n \rightarrow +\infty} \frac{|\mathcal{H} \cap F_n|}{|F_n|} > \delta \quad \text{and} \quad \int_X g_1^t 1_A(x) \cdots g_l^t 1_A(x) d\mu(x) > \varepsilon \quad \forall t \in \mathcal{H}.$$

Here δ, ε depend upon $\{F_n\}_1^\infty$.

Proof. Sufficiency. Since

$$\begin{aligned} & \liminf_{n \rightarrow +\infty} \frac{1}{|F_n|} \int_{F_n} \int_X g_1^t 1_A \cdots g_l^t 1_A d\mu dt \\ & \geq \liminf_{n \rightarrow +\infty} \frac{|\mathcal{H} \cap F_n|}{|F_n|} \frac{1}{|F_n \cap \mathcal{H}|} \int_{F_n \cap \mathcal{H}} \int_X g_1^t 1_A \cdots g_l^t 1_A d\mu dt \\ & \geq \delta \varepsilon, \end{aligned}$$

the sufficiency holds.

Necessity. Take $\varepsilon > 0$ so that

$$\liminf_{n \rightarrow +\infty} \frac{1}{|F_n|} \int_{F_n} \int_X g_1^t 1_A(x) \cdots g_l^t 1_A(x) d\mu(x) dt > 3\varepsilon.$$

Set

$$\mathcal{H} = \left\{ t \in R \mid \int_X g_1^t 1_A(x) \cdots g_l^t 1_A(x) d\mu(x) > \varepsilon \right\} \quad \text{and} \quad \mathcal{H}^c = R \setminus \mathcal{H}.$$

Clearly \mathcal{H} is a Borel subset of R . To prove the necessity, we only need to prove that $D_*(\mathcal{H}) > 0$ over $\{F_n\}_1^\infty$. Indeed, otherwise,

$$\begin{aligned} 3\varepsilon & < \liminf_{n \rightarrow +\infty} \frac{1}{|F_n|} \int_{F_n} \int_X g_1^t 1_A(x) \cdots g_l^t 1_A(x) d\mu(x) dt \\ & = \liminf_{n \rightarrow +\infty} \frac{|\mathcal{H} \cap F_n|}{|F_n|} \frac{1}{|\mathcal{H} \cap F_n|} \int_{F_n \cap \mathcal{H}} \int_X g_1^t 1_A(x) \cdots g_l^t 1_A(x) d\mu(x) dt \\ & \quad + \limsup_{n \rightarrow +\infty} \frac{1}{|F_n|} \int_{F_n \cap \mathcal{H}^c} \int_X g_1^t 1_A(x) \cdots g_l^t 1_A(x) d\mu(x) dt \\ & \leq \limsup_{n \rightarrow +\infty} \frac{1}{|F_n|} \cdot \varepsilon |F_n \cap \mathcal{H}^c| \\ & \leq \varepsilon \end{aligned}$$

which is a contradiction.

The proof of Lemma 1.2 is thus completed. \square

The following is useful for us to prove the multiple recurrence theorem (Theorem 0.1) by transfinite induction.

Proposition 1.3. *Let $\{\mathcal{X}_\theta; \theta \in \Theta\}$ be a totally ordered family of σ -subalgebras of \mathcal{X} ; that is to say, for any $\theta_1, \theta_2 \in \Theta$ either $\mathcal{X}_{\theta_1} \subset \mathcal{X}_{\theta_2}$ or $\mathcal{X}_{\theta_2} \subset \mathcal{X}_{\theta_1}$. If each $G \curvearrowright_T (X, \mathcal{X}_\theta, \mu)$ is an Sz-system and set $\mathcal{Y} = \sigma(\bigcup_\theta \mathcal{X}_\theta)$, then $G \curvearrowright_T (X, \mathcal{Y}, \mu)$ is an Sz-system.*

Proof. Here our argument is different with and more concise than the proof of [23, Proposition 7.1].³ Let $A \in \mathcal{Y}$ with $\mu(A) > 0$ and let $g_1, \dots, g_l \in G$. Since $\bigcup_\theta \mathcal{X}_\theta$ is an algebra generating \mathcal{Y} , by induction we can find $A_n \in \bigcup_\theta \mathcal{X}_\theta$ and $\epsilon_n \downarrow 0$ so that

$$\mu(A \triangle A_n) < (2^{-l}\epsilon_n)^2 \quad \text{and} \quad A_n \subseteq A_{n+1}, \quad n = 1, 2, \dots$$

Then $\|1_A - 1_{A_n}\|_2 < \epsilon_n 2^{-l}$ in $\mathcal{L}^2(X, \mathcal{Y}, \mu)$ for each n .

Given any asymptotic Følner sequence $\{F_n\}_1^\infty$ in $(R, +)$ and $i_0 \geq 1$, by Lemma 1.2 it follows that for some $\varepsilon > 0$ and $\delta > 0$, the set

$$\mathcal{H}_{i_0} = \left\{ t \in R \left| \int_X g_1^t 1_{A_{i_0}} \cdots g_l^t 1_{A_{i_0}} d\mu > \varepsilon \right. \right\}$$

has lower density $\geq \delta$ over $\{F_n\}_1^\infty$. Since $\{A_n\}$ is monotonically increasing,

$$\mathcal{H}_i = \left\{ t \in R \left| \int_X g_1^t 1_{A_i} \cdots g_l^t 1_{A_i} d\mu > \varepsilon \right. \right\} \supseteq \mathcal{H}_{i_0} \quad \forall i \geq i_0.$$

Now by

$$1_A(x) = 1_{A_i}(x) + \psi_i(x) \quad \text{where } \|\psi_i\|_\infty \leq 1 \text{ and } \|\psi_i\|_2 < 2^{-l}\epsilon_i$$

we can obtain that

$$\begin{aligned} \int_X g_1^t 1_A(x) \cdots g_l^t 1_A(x) d\mu(x) &= \int_X g_1^t (1_{A_i} + \psi_i) \cdots g_l^t (1_{A_i} + \psi_i) d\mu \\ &\geq \int_X g_1^t 1_{A_i} \cdots g_l^t 1_{A_i} d\mu - \epsilon_i \\ &> \varepsilon - \epsilon_i \quad \forall t \in \mathcal{H}_i, i \geq i_0. \end{aligned}$$

Thus, by Lemma 1.2 once again, it follows that $G \curvearrowright_T (X, \mathcal{Y}, \mu)$ is an Sz-system.

This proves Proposition 1.3. \square

This result immediately leads to the following important fact, which is one of our three ladders for proving Theorem 0.1.

Corollary 1.4. *Let $\{X_\xi; \xi \leq \eta\}$ be a Furstenberg factors chain of a nontrivial standard Borel G -system $X = G \curvearrowright_T (X, \mathcal{X}, \mu)$. Assume that the ordinal ξ is a limit ordinal and that X_θ is an Sz-system for each $\theta < \xi$. Then X_ξ is an Sz-system.*

Proof. According to Structure Theorem, $\{\mathcal{X}_\theta; \theta < \xi\}$ is totally ordered. Then the statement follows from the foregoing Proposition 1.3. \square

By an argument similar to that of Corollary 1.4, we can easily obtain the following

Lemma 1.5. *Let $\{X_\xi; \xi \leq \eta\}$ be a Furstenberg factors chain of a nontrivial standard Borel G -system $X = G \curvearrowright_T (X, \mathcal{X}, \mu)$. Assume that the ordinal ξ is a limit ordinal and that X_θ is a weak Sz-system for each $\theta < \xi$. Then X_ξ is a weak Sz-system.*

³Inasmuch as the original proof of [23, Proposition 7.1] for $R = \mathbb{Z}$ (also see [16, Proposition 7.26] for $G = \mathbb{Z}$) involves the disintegration $\{\mu_x; x \in X\}$ of μ given \mathcal{X}_θ , whence there (X, \mathcal{X}, μ) had to be a standard Borel space to obtain a disintegration of $\mu: \mu = \int_X \mu_x d\mu$ over \mathcal{X}_θ in [24, 23, 16].

2. Sz-property of totally relatively weak-mixing extensions

Let $(G, +)$ be an lcsc R -module with the zero element \mathbf{o} and with the continuous scalar multiplication $(t, g) \mapsto tg$ of $R \times G$ to G . Let there be any given a short factors series:

$$X = G \curvearrowright_T (X, \mathcal{X}, \mu) \xrightarrow{Id_X} X' = G \curvearrowright_T (X, \mathcal{X}', \mu) \xrightarrow{\pi} Y = G \curvearrowright_S (Y, \mathcal{Y}, \nu),$$

where (X, \mathcal{X}, μ) is a standard Borel G -space so we can decompose $\mu = \int_Y \mu_y d\nu(y)$ over (Y, π) .

By D-lim we denote the limit in density (cf. [23, 15]). The following lemma is [15, Proposition 3.7].

Lemma 2.1 ([15]). *Let $\pi: X' \rightarrow Y$ be totally relatively weak-mixing for G and let $g_1, \dots, g_l \in G$ with $g_i \neq \mathbf{o}$ and $g_i \neq g_j$ for $1 \leq i \neq j \leq l$. If $f_1, \dots, f_l \in \mathcal{L}^\infty(X, \mathcal{X}', \mu)$, then in the weak topology of $\mathcal{L}^2(X, \mathcal{X}', \mu)$, it holds that*

$$\text{D-lim}_{t \in R} \left\{ \prod_{i=1}^l T_{g_i}^t f_i - \prod_{i=1}^l T_{g_i}^t E_\mu(f_i | \pi^{-1}[\mathcal{Y}]) \right\} = 0$$

over any weak Følner sequence in $(R, +)$.

We can restate this lemma as the following more convenient version for our later arguments:

Lemma 2.2. *Let $\pi: X' \rightarrow Y$ be totally relatively weak-mixing for G and let $g_1, \dots, g_l \in G$ with $g_i \neq g_j$ for $1 \leq i \neq j \leq l$. If $\psi_1, \dots, \psi_l \in \mathcal{L}^\infty(X, \mathcal{X}', \mu)$, then in the weak topology of $\mathcal{L}^2(X, \mathcal{X}', \mu)$, it holds that*

$$\text{D-lim}_{t \in R} \left\{ \prod_{i=1}^l T_{g_i}^t \psi_i - \prod_{i=1}^l T_{g_i}^t E_\mu(\psi_i | \pi^{-1}[\mathcal{Y}]) \right\} = 0$$

over any weak Følner sequence in $(R, +)$.

Proof. If $g_i \neq \mathbf{o}$ for all $1 \leq i \leq l$, then this is just Lemma 2.1. If not, replace all the g_i by some $g_0 + g_i$ where $g_0 \neq -g_i$ for any $1 \leq i \leq l$. Since μ is g_0^t -invariant for each $t \in R$, this implies the desired statement. \square

This lemma makes it evident that a totally relatively weak-mixing extension of an Sz-factor is an Sz-system. That is the following

Corollary 2.3. *If $X' = G \curvearrowright_T (X, \mathcal{X}', \mu)$ is a totally relatively weak-mixing extension of an (resp. a weak) Sz-system $Y = G \curvearrowright_S (Y, \mathcal{Y}, \nu)$, then X' is an (resp. a weak) Sz-system.*

Proof. This follows immediately from Def. 1.1, Lemma 1.2 and Lemma 2.2. Indeed, given any $A \in \mathcal{X}'$ with $\mu(A) > 0$ and any distinct elements $g_1, \dots, g_l \in G$, over any asymptotic Følner sequence $\{F_n\}_1^\infty$ in $(R, +)$ we can take $\delta > 0, \varepsilon > 0$ such that

$$\left\{ t \in R: \int_Y S_{g_1}^t E_\mu(1_A | Y) \cdots S_{g_l}^t E_\mu(1_A | Y) d\nu > 2\varepsilon \right\}$$

has lower density $\geq \delta$ over $\{F_n\}_1^\infty$. Then by Lemma 2.2, it follows that

$$\left\{ t \in R: \int_X T_{g_1}^t 1_A \cdots T_{g_l}^t 1_A d\mu > \varepsilon \right\}$$

has lower density $\geq \delta$ over $\{F_n\}_1^\infty$. If $g_1 = g_2$ then $T_{g_1}^t 1_A T_{g_2}^t 1_A = T_{g_1}^t 1_A$ and so the above argument is still valid for any $g_1, \dots, g_l \in G$. This proves Corollary 2.3. \square

Another important consequence of Lemma 2.2 is the following, which is a generalization of [23, Lemma 7.9].

Corollary 2.4. *Let $\mathbf{X}' = G \curvearrowright_T (X, \mathcal{X}', \mu)$ be a totally relatively weak-mixing extension of the G -system $\mathbf{Y} = G \curvearrowright_S (Y, \mathcal{Y}, \nu)$ and let $g_1, \dots, g_l \in G$ with $g_i \neq g_j$ for $1 \leq i \neq j \leq l$. Let $\psi_1, \dots, \psi_l \in \mathcal{L}^\infty(X, \mathcal{X}', \mu)$. Then for any $\delta > 0$ and $\varepsilon > 0$,*

$$\left\{ t \in R \mid \nu \left\{ y \in Y : \left| \int_X \prod_{i=1}^l T_{g_i}^t \psi_i d\mu_y - \prod_{i=1}^l E_\mu(\psi_i | \mathbf{Y})(S_{g_i}^t y) \right| > \varepsilon \right\} > \delta \right\}$$

is of density 0 over any weak Følner sequence $\{F_n\}_1^\infty$ in $(R, +)$.

3. Primitive extensions

Let $(G, +)$ be an lcsc R -module if no an explicit declaration. As before, write $T_{tg} = T_g^t = g^t$ and $T_{g+h} = g \circ h$ for any $t \in R$ and $g, h \in G$. Let

$$\mathbf{X} = G \curvearrowright_T (X, \mathcal{X}, \mu) \xrightarrow{Id_X} \mathbf{X}' = G \curvearrowright_T (X, \mathcal{X}', \mu) \xrightarrow{\pi} \mathbf{Y} = G \curvearrowright_S (Y, \mathcal{Y}, \nu)$$

be any short factors series, where (X, \mathcal{X}, μ) is a standard Borel probability space. Let

$$\mu = \int_Y \mu_y d\nu(y)$$

be the standard disintegration of μ , over $\pi: \mathbf{X} \rightarrow \mathbf{Y}$, and simply write $\|\cdot\|_{2,y} = \|\cdot\|_{2,\mu_y}$ for ν -a.e. $y \in Y$. As usual we shall say $\pi: \mathbf{X}' \rightarrow \mathbf{Y}$ is **primitive** if there exists direct sum

$$G = G_{rc} \oplus G_{rw}$$

of nontrivial R -submodules of G so that $\pi: \mathbf{X}' \rightarrow \mathbf{Y}$ is relatively compact for G_{rc} and totally relatively weak-mixing for G_{rw} . Particularly, if $G_{rc} = G$, we say $\pi: \mathbf{X}' \rightarrow \mathbf{Y}$ is a **relatively compact** extension of \mathbf{Y} . See [24, 23] and [14, §4].

We need a finitary version of the van der Waerden theorem for any R -module as follows.

Lemma 3.1 ([13]). *Let $G_1 \subset G_2 \subset \dots \subset G_n \subset \dots$ be a sequence subsets of an R -semimodule $(G, +)$ with $G = \bigcup_n G_n$; and let F be a finite set of G and $q \in \mathbb{N}$. Then there exists a number $N = N(q, F)$ such that whenever $n \geq N$ and $G_n = B_1 \cup \dots \cup B_q$ is a partition of G into q sets, one of these B_j contains a homothetic copy of F , $\{g_0 + rf \mid f \in F\}$, where $g_0 \in G$ and $r \in R$ with $r \neq 0$.*

Given any $g \in G$ and $m \in \mathbb{N}$, set $mg = \overbrace{(1 + \dots + 1)}^{m\text{-times}} g$, where 1 is the identity element of $(R, +, \cdot)$. For our convenience we now introduce a notation.

Definition 3.2. We shall say an R -module G is **irreducible** relative to a subset F of G provided that if $mf \neq o$ for any nonzero $f \in F$ and any $m \in \mathbb{N}$.

Lemma 3.1 leads to the following useful result, which generalizes [23, Lemma 7.11].

Corollary 3.3. *Let $K \in \mathbb{N}$ and $F = \{g_1, g_2, \dots, g_l\} \subset G$ be any given, where G is an irreducible R -module relative to F . Then there is a finite subset $Q \subset G$ and an integer $M \geq 1$, such that for any coloring map*

$$k: G \rightarrow \{1, 2, \dots, K\}$$

there exists some $g' \in Q$ and some $m \in \mathbb{N}$, $1 \leq m \leq M$, such that

$$k(g' + mg_1) = k(g' + mg_2) = \dots = k(g' + mg_l).$$

Proof. First we can choose a countable dense subgroup G' of G with $F \subset G'$. We will regard it as a \mathbb{Z}_+ -semimodule.

According to Lemma 3.1 for $F = \{g_1, \dots, g_l\} \subset G'$ and $q = K$, there exists a finite subset G'_L of G' such that for any coloring map $k: G'_L \rightarrow \{1, 2, \dots, K\}$, k will be constant on a subset of G'_L of the form $\{g_0 + mg_1, g_0 + mg_2, \dots, g_0 + mg_l\}$ where $m \neq 0$. But there are only finitely many possibilities for $m \in \mathbb{Z}_+$ and $g_0 \in G'$ with $g_0 + mg_1 \in G'_L$ and $g_0 + mg_2 \in G'_L$, assuming, as we may, that $g_1 \neq g_2$. Carrying this over to G' and then to G , we can conclude the statement. \square

The implication of the relatively compactness of $\pi: G_{rc} \curvearrowright_T (X, \mathcal{X}', \mu) \rightarrow G_{rc} \curvearrowright_S (Y, \mathcal{Y}, \nu)$ is summarized in the following lemma, which generalizes [23, Lemma 7.10].

Lemma 3.4. *Let $\pi: G_{rc} \curvearrowright_T (X, \mathcal{X}', \mu) \rightarrow G_{rc} \curvearrowright_S (Y, \mathcal{Y}, \nu)$ be relatively compact. Let $A \in \mathcal{X}'$ with $\mu(A) > 0$. Then we can find $A' \subseteq A$ with $\mu(A')$ as close as we like to $\mu(A)$ having the property: For any $\varepsilon > 0$ there exists a finite set of functions ϕ_1, \dots, ϕ_K in $\mathcal{L}^2(X, \mathcal{X}', \mu)$ and a function*

$$k: Y \times G_{rc} \rightarrow \{1, 2, \dots, K\}$$

such that for every $h \in G_{rc}$, $\|T_h 1_{A'} - \phi_{k(y,h)}\|_{2,\nu}$ for ν -a.e. $y \in Y$.

Proof. This result follows easily from the characterization theorem of relatively compact extensions [14, Theorem 2.4]. \square

Now we are ready to prove the following result which is one of our ladders for proving the multiple recurrence theorem.

Proposition 3.5. *Let $(G, +)$ be an irreducible lcsc R -module. If $\pi: X' \rightarrow Y$ is a primitive extension of an Sz-system Y over asymptotic Følner sequences in $(R, +)$, then X' is also an Sz-system over asymptotic Følner sequences in $(R, +)$.*

Proof. Our proof will follow from the framework of arguing of Furstenberg [23, Proposition 7.12].

In what follows, let $A \in \mathcal{X}'$ with $\mu(A) > 0$, and let $g_1, \dots, g_l \in G$ be any given.

In view of Lemma 3.4, we may assume that 1_A is FK a.p. for G_{rc} described in Lemma 3.4. Writing $\mu(A) = \int_Y \mu_y(A) d\nu(y)$, we see that if we set $a = \frac{1}{2}\mu(A)$, there exists a set $B \in \mathcal{Y}$ with $\nu(B) > 0$ such that $\mu_y(A) \geq a$ for every $y \in B$. We express the given elements $g_1, \dots, g_l \in G$ as follows:

$$g_1 = h_1 + f_1, g_2 = h_2 + f_2, \dots, g_l = h_l + f_l, \quad \text{where } h_i \in G_{rc} \text{ and } f_i \in G_{rw} \text{ for } 1 \leq i \leq l.$$

We let $\{F_n\}_1^\infty$ be any given asymptotic Følner sequence in $(R, +)$.

Let $a_1 < a^l$. We shall show that there exists a measurable set $P \subset R$ with positive lower density over $\{F_n\}_1^\infty$ and $\eta > 0$ such that for each $t \in P$ there is a set $B_t \in \mathcal{Y}$ with $\nu(B_t) > \eta$ such that

$$\mu_y \left(\bigcap_{i=1}^l T_{h_i}^{-t} T_{f_i}^{-t} A \right) > a_1 \quad \forall y \in B_t. \quad (3.1)$$

This will implies Proposition 3.5, since this leads to

$$\mu \left(\bigcap_{i=1}^l T_{g_i}^{-t} A \right) > a_1 \eta \quad \forall t \in P.$$

The set B_t will be defined by two requirements. For $a_1 < a_2 < a^l$ we shall require

$$\mu_y \left(\bigcap_{i=1}^l T_{f_i}^{-t} A \right) > a_2 \quad \forall y \in B_t \text{ with } t \in P. \quad (3.2)$$

Secondly we prescribe $\varepsilon_1 > 0$ such that if $\mu_y \left(T_{f_i}^{-t} T_{h_i}^{-t} A \triangle T_{f_i}^{-t} A \right) < \varepsilon_1$ for all $1 \leq i \leq l$, then (3.2) implies (3.1). Then we require

$$\mu_y \left(T_{f_i}^{-t} T_{h_i}^{-t} A \triangle T_{f_i}^{-t} A \right) < \varepsilon_1, \quad 1 \leq i \leq l, \quad (3.3)$$

when $t \in P$ and $y \in B_t$.

Suppose now that P and $\{B_t; t \in P\}$ have been found so that (3.3) is satisfied and, in addition,

$$S_{f_i}^t y \in B, \quad \forall y \in B_t, \quad 1 \leq i \leq l. \quad (3.4)$$

Apply Corollary 2.4 with $\psi_1 = \dots = \psi_l = 1_A$, $\varepsilon < a^l - a_2$ and $\delta < \frac{1}{2} \min_{t \in P} \nu(B_t)$. Then

$$\mu_y \left(\bigcap_{i=1}^l T_{f_i}^{-t} A \right) = \int_X \prod_{i=1}^l T_{f_i}^t f d\mu_y > \prod_{i=1}^l S_{f_i}^t E_\mu(1_A | Y)(y) - \varepsilon = \prod_{i=1}^l \mu_{S_{f_i}^t y}(A) - \varepsilon \geq a^l - \varepsilon > a_2$$

for any $y \in B_t$ but for a set of y of measure $< \frac{1}{2} \nu(B_t)$ and for $t \in P$ outside a set of density 0. Modifying P and B_t accordingly, we will be left with a set—call it again P —with positive lower density, and for each $t \in P$ a set—call it again B_t —with $\inf_{t \in P} \nu(B_t) > 0$ such that for these t and y , (3.2) and (3.3) are valid. As we have seen, (3.3) + (3.4) \Rightarrow (3.3) + (3.2) \Rightarrow (3.1). Thus the problem is reduced to finding P and $\{B_t; t \in P\}$ such that (3.3) and (3.4) are satisfied.

Let $\varepsilon_2 < \frac{1}{2} \sqrt{\varepsilon_1}$ be any given. Recall that 1_A is FK a.p. for G_{rc} . By Lemma 3.4, we can then find functions $\phi_1, \dots, \phi_K \in \mathcal{L}^2(X, \mathcal{X}, \mu)$ and a measurable coloring map

$$k: Y \times G_{rc} \rightarrow \{1, 2, \dots, K\}$$

such that

$$\|T_h 1_A - \phi_{k(y,h)}\|_{2,y} < \varepsilon_2 \quad \forall h \in G_{rc} \text{ and } \nu\text{-a.e. } y \in Y.$$

We now define a family of maps

$$\{k_r: Y \times G \rightarrow \{1, 2, \dots, K\}\}_{r \in R}$$

by

$$k_r(y, h \mathbin{+} f) = k(S_f^r y, rh) \quad \forall (h, f) \in G = G_{rc} \oplus G_{rw}.$$

We then have that for any $r \in R$, for ν -a.e. $y \in Y$,

$$\|S_f^r T_h^r 1_A - S_f^r \phi_{k_r(y, h \mathbin{+} f)}\|_{2, y} = \|T_h^r 1_A - \phi_{k(S_f^r y, rh)}\|_{2, S_f^r y} < \varepsilon_2. \quad (3.5)$$

Fix $r \in R$ and $y \in Y$ and apply Corollary 3.3 to the map $k_r(y, \cdot)$ on $G = G_{rc} \oplus G_{rw}$ with the integer K and finite set $F = \{f_1, \dots, f_l, g_1, \dots, g_l\}$. Independently of r and y there is a finite subset $Q \subset G$ and an integer $M \geq 1$ such that $k_r(y, g' \mathbin{+} mg_i)$ and $k_r(y, g' \mathbin{+} mf_i)$ both take on the same value for $1 \leq i \leq l$ for some $g' \in Q$ and some m with $1 \leq m \leq M$. If ℓ is this value, we write $\phi_{(r, y)} = \phi_\ell$. Then if $g' = h' \mathbin{+} f'$ and simply write $T_g^t = g^t$ and $S_g^t = g^t$, then

$$\begin{aligned} \|f_i^{mr} h_i^{mr} 1_A - f_i^{mr} (h'^{-r} \phi_{(r, y)})\|_{2, g^r y} &= \|g'^r f_i^{mr} h_i^{mr} 1_A - f'^r f_i^{mr} \phi_{(r, y)}\|_{2, y} \\ &= \|f'^r f_i^{mr} h'^r h_i^{mr} 1_A - f'^r f_i^{mr} \phi_{(r, y)}\|_{2, y} \\ &< \varepsilon_2 \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} \|f_i^{mr} 1_A - f_i^{mr} h'^{-r} \phi_{(r, y)}\|_{2, g^r y} &= \|g'^r f_i^{mr} 1_A - f'^r f_i^{mr} \phi_{(r, y)}\|_{2, y} \\ &= \|f'^r f_i^{mr} h'^r 1_A - f'^r f_i^{mr} \phi_{(r, y)}\|_{2, y} \\ &< \varepsilon_2 \end{aligned} \quad (3.7)$$

for all $1 \leq i \leq l$ by (3.5), and moreover

$$\begin{aligned} \|1_A - h'^{-r} \phi_{(r, y)}\|_{2, g^r y} &= \|g'^r 1_A - f'^r \phi_{(r, y)}\|_{2, y} \\ &= \|h'^r 1_A - \phi_{k(f'^r y, rh')}\|_{2, y} \\ &< \varepsilon_2. \end{aligned} \quad (3.8)$$

What we have shown is that for every $r \in R$ and ν -a.e. $y \in Y$, there exist m and g' , both having a finite range of possibilities, such that

$$\|f_i^{mr} h_i^{mr} 1_A - f_i^{mr} 1_A\|_{2, g^r y} < \sqrt{\varepsilon_1} \quad (1 \leq i \leq l). \quad (3.9)$$

We are now ready to produce the set P and the associated sets $B_t, t \in P$ such that both (3.3) and (3.4) hold for $(y, t), y \in B_t$.

For each $r \in R$ we form the set

$$C_r = \bigcap_{i, m, g'} (f_i^m \circ g')^{-r} B \in \mathcal{Y} \quad (3.10)$$

where the intersection is taken over i, m, g' with $1 \leq i \leq l, 1 \leq m \leq M$ and $g' \in Q$. Here we use the fact that $Y = G \curvearrowright_S (Y, \mathcal{Y}, \nu)$ is an Sz-system. It follows from Lemma 1.2 that there exists a subset P' of R such that P' is of positive lower density over $\{F_n\}_1^\infty$ and $\nu(C_r) > \eta' > 0$ for each $r \in P'$. Now let $y \in C_r$ for $r \in P'$. There exists $m = m(r, y)$ and $g' = g'(r, y)$ such that (3.9) holds and $f_i^{mr}(g'^r y) \in B$ for $1 \leq i \leq l$. Let J be the total number of possibilities for (m, g') . Then since

$y \mapsto \mu_y$ is measurable, hence for a \mathcal{Y} -set $D_r \subset C_r$ with $\nu(D_r) > \frac{1}{J}\eta'$, $m(r, y)$ and $g'(r, y)$ take on a constant value, say $m(r)$, $g(r)$. We now define $t(r) = m(r)r$, and set $P = \{t(r); r \in P'\}$ and

$$B_{t(r)} = g(r)^{r'} D_r \quad (= S_{g(r)}^r D_r) \quad \forall r \in P'. \quad (3.11)$$

Then for any $t = t(r) \in P$, we have

$$\nu(B_t) > \frac{1}{J}\eta', \quad f_i^t B_t \subset B, \quad \|f_i^t h_i^t 1_A - f_i^t 1_A\|_{2,y} < \varepsilon_2 \quad (1 \leq i \leq l) \quad (3.12)$$

for any $y \in B_t$.

Finally we need to show that P is of positive lower density over $\{F_n\}_1^\infty$. Indeed, let $\{F'_n\}_1^\infty$ be any subsequence of $\{F_n\}_1^\infty$. Set

$$P'_1 = \{r \in P' \mid m(r) = 1\}, \dots, P'_M = \{r \in P' \mid m(r) = M\}.$$

Then by

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \frac{|P' \cap F'_n|}{|F'_n|} &= \liminf_{n \rightarrow +\infty} \frac{|P'_1 \cap F'_n| + \dots + |P'_M \cap F'_n|}{|F'_n|} \\ &= \liminf_{n \rightarrow +\infty} \frac{|(1P'_1) \cap (1F'_n)| + \dots + |(MP'_M) \cap (MF'_n)|}{|F'_n|} \\ &\leq \sum_{m=1}^M \limsup_{n \rightarrow +\infty} \frac{|P \cap (mF'_n)|}{|F'_n|} \\ &\leq \sum_{m=1}^M \limsup_{n \rightarrow +\infty} \frac{c_m |P \cap F'_{mn}|}{|F'_{mn}|} \\ &\leq (c_1 + \dots + c_M) \limsup_{n \rightarrow +\infty} \frac{|P \cap F'_n|}{|F'_n|} \end{aligned}$$

we see that

$$\limsup_{n \rightarrow +\infty} \frac{|P \cap F'_n|}{|F'_n|} > 0.$$

This implies that

$$\liminf_{n \rightarrow +\infty} \frac{|P \cap F_n|}{|F_n|} > 0.$$

Now we have fulfilled all of our requirements, and thus this concludes the proof of Proposition 3.5. \square

The following result is contained in the proof of Proposition 3.5.

Proposition 3.6. *Let $(G, +)$ be an lcsc R -module. If $\pi: X' \rightarrow Y$ is a primitive extension of the system Y which satisfies that for any $B \in \mathcal{Y}$ with $\nu(B) > 0$ and any $g_1, \dots, g_l \in G$ there exists some integer $N = N(B, g_1, \dots, g_l) \geq 1$ such that for any weak Følner sequence $\{F_n\}_1^\infty$ in $(R, +)$*

$$\limsup_{n \rightarrow +\infty} \frac{1}{|F_n|} \int_{F_n} \nu(B \cap S_{g_1}^{-m't} B \cap \dots \cap S_{g_l}^{-m't} B) dt > 0 \quad (3.13)$$

for some integer m' with $1 \leq m' \leq N$, then X' also possesses this property.

Proof. We first note that there is no loss of generality in assuming that G is irreducible relative to the finite set $F = \{g_1, \dots, g_l\} \subset G$. Let $A \in \mathcal{X}$ with $\mu(A) > 0$ and then $B \in \mathcal{Y}$ with $\nu(B) > 0$ be as in the proof of Proposition 3.5. Similar to (3.10), for each $t \in R$ we form the set

$$C_t = \bigcap_{i,m,g'} (f_i^m \circ g')^{-m't} B \in \mathcal{Y} \quad (3.10)'$$

for some integer $m' \geq 1$, where the intersection is taken over i, m, g' with $1 \leq i \leq l, 1 \leq m \leq M$ and $g' \in Q$. Here we use the fact that $Y = G \curvearrowright_S (Y, \mathcal{Y}, \nu)$ satisfies (3.13) with the set

$$F_B := \{m' m f_i + m' g' \mid 1 \leq i \leq l, 1 \leq m \leq M, g' \in Q\}$$

in place of F . Then for any weak Følner sequence $\{F_n\}_1^\infty$ in $(R, +)$ there exists an integer m' with $1 \leq m' \leq N(B, F_B)$ and a subset P of R such that P is of positive upper density and $\nu(C_t) > \eta' > 0$ for each $t \in P$, where η' relies on $\{F_n\}_1^\infty$.

Now let $y \in C_t$ for $t \in P$. There exists $m = m(t, y)$ and $g' = g'(t, y)$ such that

$$\|f_i^{m'mt} h_i^{m'mt} 1_A - f_i^{m'mt} 1_A\|_{2, g^{m't} y} < \sqrt{\varepsilon_1} \text{ and } f_i^{m'mt}(g^{m't} y) \in B \quad (1 \leq i \leq l). \quad (3.9)'$$

for all $1 \leq i \leq l$. Let J be the total number of possibilities for $(m, g') \in \{1, 2, \dots, M\} \times Q$. Hence for a \mathcal{Y} -set $D_t \subset C_t$ with $\nu(D_t) > \frac{1}{J}\eta'$, $m(t, y)$ and $g'(t, y)$ take on a constant value, say m_t, g'_t . We now define

$$B_t = g_t^{m't} D_t \quad (= S_{g'_t}^{m't} D_t) \quad \forall t \in P. \quad (3.11)'$$

Then for any $t \in P$, we have

$$\nu(B_t) > \frac{1}{J}\eta', \quad f_i^{m_t m't} B_t \subset B, \quad \|f_i^{m_t m't} h_i^{m_t m't} 1_A - f_i^{m_t m't} 1_A\|_{2, y} < \varepsilon_2 \quad (1 \leq i \leq l) \quad (3.12)'$$

for any $y \in B_t$. Set

$$P_1 = \{t \in P \mid m_t = 1\}, \dots, P_M = \{t \in P \mid m_t = M\}.$$

Then there is some integer k with $1 \leq k \leq M$ such that

$$\limsup_{n \rightarrow +\infty} \frac{|P_k \cap F_n|}{|F_n|} > 0.$$

This implies that

$$\limsup_{n \rightarrow +\infty} \frac{1}{|F_n|} \int_{F_n} \mu(A \cap g_1^{-km't} A \cap \dots \cap g_l^{-km't} A) dt > 0.$$

Now noting that $1 \leq km' \leq M \cdot N(B, F_B)$ and that $M \cdot N(B, F_B)$ is independent of $\{F_n\}_1^\infty$, this thus concludes the proof of Proposition 3.6. \square

4. The measure-theoretic multiple recurrence

This section will be devoted to proving the multiple recurrence theorems, namely Theorem 0.1 and 0.3, stated in Introduction.

4.1. Proof of Theorem 0.1

Let $\pi: X = (X, \mathcal{X}, \mu, G) \rightarrow Y = (Y, \mathcal{Y}, \nu, G)$ be a standard Borel extension of a factor Y , where G is an irreducible lcscN module over a syndetic ring $(R, +, \cdot)$. Then the following proposition is a slightly general statement than Theorem 0.1.

Proposition 4.1. *If Y is an Sz-factor of X , then X is an Sz-system, over any asymptotic Følner sequence in $(R, +)$.*

Proof. Let

$$X \rightarrow X_\eta \rightarrow \cdots \rightarrow X_{\xi+1} \xrightarrow{\pi_{\xi+1, \xi}} X_\xi \rightarrow \cdots \rightarrow X_2 \xrightarrow{\pi_{2,1}} X_1 \xrightarrow{\pi_{1,0}} Y$$

be the Furstenberg factors chain of $X = G \curvearrowright_T (X, \mathcal{X}, \mu)$ starting from Y . Since Y is an Sz-system over any asymptotic Følner sequence in $(R, +)$, hence from Corollaries 1.4, 2.3 and Proposition 3.5 we can see that X is an Sz-system by transfinite induction. This thus concludes the proof of Proposition 4.1. \square

4.2. Proof of Theorem 0.3

Based on Furstenberg Structure Theorem, by transfinite induction, Theorem 0.3 follows immediately from Lemma 1.5, Corollary 2.3 and Proposition 3.6.

5. Multiple Birkhoff center and pointwise multirecurrence

Let X be a compact metric space X in the sequel. In this section, the topological dynamical system (for brevity, *t.d.s.*) $T: G \times X \rightarrow X$ or $G \curvearrowright_T X$ we now consider is as in §0.3, where G is an lcscN module over a syndetic ring $(R, +, \cdot)$ with a Haar measure $|\cdot|$ (or write dt). In addition, assume R is not compact, i.e., $|R| = \infty$.

5.1. Poisson stable motions

Given any sequence $(T_n)_1^\infty$ of elements of R , for convenience, we shall say that $T_n \rightarrow +\infty$ as $n \rightarrow +\infty$ if for any compacta $K \subset$ of R there exists a positive integer $L = L(K)$ for which

$$n \geq L \Rightarrow T_n \notin K.$$

It will be a useful fact that if $T_n \rightarrow +\infty$ in R then $t + T_n \rightarrow +\infty$ for every $t \in R$. Since there does not need to have any order in R , the above notion “ $T_n \rightarrow +\infty$ ” ought to be of interest.

According to [37, Def. 4.15], a sequence $\{R_n\}_1^\infty$ of nonnull compact subsets of R is called a *summing sequence* in $(R, +)$ if the following conditions are satisfied:

- (1) $R = \bigcup_1^\infty R_n$;
- (2) $R_n \subset \text{Int}(R_{n+1}) \forall n \geq 1$, where $\text{Int}(\cdot)$ denotes the interior of a set;
- (3) $|(r + R_n) \triangle R_n| \cdot |R_n|^{-1} \rightarrow 0$ as $n \rightarrow +\infty$, for any $r \in R$ and then uniformly for r in any compacta of R .

Since $(R, +)$ is an abelian, lcsc, and Hausdorff group here, it is amenable and σ -compact. Therefore, there always exists a summing sequence in $(R, +)$; see e.g. [37, Theorem 4.16]. Clearly a summing sequence is a weak Følner sequence by condition (3). From now on,

- let $\mathcal{R} = \{R_n\}_1^\infty$ be an arbitrarily fixed summing sequence in $(R, +)$.

For a sequence $(T_n)_{n=1}^\infty$ of elements of R , we say $T_n \xrightarrow{\mathcal{R}} +\infty$ as $n \rightarrow +\infty$ if for any integer $n \geq 1$ there is some $L = L(n) > 1$ such that $T_\ell \notin R_n$ as $\ell \geq L$. Then the following fact is obvious.

Lemma 5.1. $T_n \rightarrow +\infty$ in R if and only if $T_n \xrightarrow{\mathcal{R}} +\infty$. Thus if $T_n \xrightarrow{\mathcal{R}} +\infty$ then $t + T_n \xrightarrow{\mathcal{R}} +\infty$ for every $t \in R$.

Proof. This follows from the fact that every compact subset K of R must be contained by R_n as n sufficiently big. \square

Next, following Furstenberg's idea [23, Chap. 2] we now introduce a basic concept – multiple recurrent motion for G -action system as follows:

Definition 5.2. For $G \curvearrowright_T X$ a point $p \in X$ is said to be *multiply recurrent* (or *multiply Poisson stable*) if for any $l \geq 2$ and any sample elements $g_1, \dots, g_l \in G$, one can find a sequence of times $t_n \rightarrow +\infty$ in R so that $g_i^{t_n} p \rightarrow p$ as $n \rightarrow +\infty$, simultaneously for $i = 1, \dots, l$.

5.2. Multiple recurrence

Although a multiply recurrent point of $G \curvearrowright_T X$ does not need to lie in the multiple Birkhoff center $\Omega_\gamma(T)$ according to Def. 0.10 and Def. 5.2, yet the structure of $\Omega_\gamma(T)$ is made clear by the multiple recurrence via the following two lemmas.

Lemma 5.3. Given any l elements g_1, \dots, g_l in G , the (g_1, \dots, g_l) -multiple recurrent points of $G \curvearrowright_T X$ are everywhere dense in $\Omega_\gamma(T)$.

Proof. We consider the subsystem $G \curvearrowright_T \Omega_\gamma(T)$ instead of $G \curvearrowright_T X$. Let $p \in \Omega_\gamma(T)$ be any point and $\varepsilon > 0$ be an arbitrary number, and let g_1, \dots, g_l be any given l elements in G where $l \in \mathbb{N}$. It is required to prove that in the relative ε -ball $U_0 = B_\varepsilon(p)$ around p in $\Omega_\gamma(T)$ there can be found a (g_1, \dots, g_l) -recurrent point for $G \curvearrowright_T \Omega_\gamma(T)$.

Because of the regional multiple recurrence of T on $\Omega_\gamma(T)$, there can be found some time $\tau_1 \in R$, $\tau_1 \notin R_1$, simply written as $\tau_1 > R_1$, so that

$$U_0 \cap g_1^{-\tau_1} U_0 \cap \dots \cap g_l^{-\tau_1} U_0 \neq \emptyset.$$

Since the intersection of $l + 1$ open sets is still an open set, there can be found a point and a number, say $p_1 \in U_0$ and $\varepsilon_1 > 0$, such that

$$B_{\varepsilon_1}(p_1) \subset U_0 \cap g_1^{-\tau_1} U_0 \cap \dots \cap g_l^{-\tau_1} U_0.$$

We simply write $U_1 = B_{\varepsilon_1/2}(p_1)$. By virtue of the same regional multiple recurrence there can be found an element $\tau_2 > R_2$ in R with

$$U_1 \cap g_1^{-\tau_2} U_1 \cap \dots \cap g_l^{-\tau_2} U_1 \neq \emptyset$$

and there can be found a point $p_2 \in U_1$ and a number $\varepsilon_2 > 0$ such that

$$B_{\varepsilon_2}(p_2) \subset U_1 \cap g_1^{-\tau_2} U_1 \cap \dots \cap g_l^{-\tau_2} U_1.$$

Obviously, $\varepsilon_2 \leq \varepsilon_1/2$. We set $U_2 = B_{\varepsilon_2/2}(p_2)$. Next there can be found a point p_3 and a number $\varepsilon_3 > 0$ such that

$$B_{\varepsilon_3}(p_3) \subset U_2 \cap g_1^{-\tau_3} U_2 \cap \dots \cap g_l^{-\tau_3} U_2,$$

where $\tau_3 \in R$, $\tau_3 > R_3$ and $\varepsilon_3 \leq \varepsilon_2/2$.

Continuing this process without end and noting that $\overline{U}_n \subseteq U_{n-1}$ for $n = 1, 2, \dots$ and, besides, that $\text{diam}(\overline{U}_n) < \varepsilon_n \leq \varepsilon/2^{n-1}$, we obtain because of the compactness of the multiple Birkhoff center $\Omega_\gamma(T)$ a point q as the intersection of the sets U_n : $\{q\} = \bigcap_{n=1}^{\infty} U_n$. Since $g_i^{\tau_{n+1}} q \in U_n$ for $n = 1, 2, \dots$ and $i = 1, \dots, l$, we can see that $g_i^{\tau_n} q \rightarrow q$ as $n \rightarrow +\infty$ simultaneously for $i = 1, \dots, l$. Clearly $\tau_n \rightarrow +\infty$ by Lemma 5.1.

This completes the proof of Lemma 5.3. \square

We note that this result is still valid if, instead of $\Omega_\gamma(T)$, any compact set be taken which possesses the property of regional multiple recurrence relative to $(R_n)_1^\infty$.

Lemma 5.4. *Given any $g_1, \dots, g_l \in G$, the (g_1, \dots, g_l) -multiple recurrent points of $G \curvearrowright_T X$ form a residual subset of $\Omega_\gamma(T)$.*

Proof. Let $\{\varepsilon_n\}$ be a sequence of positive numbers with $\varepsilon_n \downarrow 0$ as $n \rightarrow +\infty$. Define sets

$$X_n = \left\{ p \in X \mid \max_{1 \leq i \leq l} d(p, g_i^t p) \geq \varepsilon_n \ \forall t > R_n \right\};$$

where X_n may be an empty set. Obviously, all the points $p \in X_n$ are not (g_1, \dots, g_l) -recurrent for T , and it is easy to check that every point which is not (g_1, \dots, g_l) -recurrent for T lies in some X_n . Clearly, X_n is closed by the continuity of $T(tg, x)$ with respect to $(t, g, x) \in R \times G \times X$. Furthermore, X_n is nowhere dense in $\Omega_\gamma(T)$ by Lemma 5.3. This implies that $\Omega_\gamma(T) - \bigcup_{n=1}^{\infty} X_n$ is a residual subset of $\Omega_\gamma(T)$ and the proof of Lemma 5.4 is thus completed. \square

The full probability property of multiply recurrent points of $G \curvearrowright_T X$ comes from the following theorem by using a measure-theoretic version of the proof of Lemma 5.3.

Theorem 5.5 (C. Carathéodory [11] for $G = \mathbb{R}$). *Given any l elements $g_1, \dots, g_l \in G$, the set $\mathcal{R}_{g_1, \dots, g_l}(T)$ of all the (g_1, \dots, g_l) -multiple recurrent points of $G \curvearrowright_T X$ is a full probability G_δ subset of X .*

Proof. By the proof of Lemma 5.4, it is easy to see that $\mathcal{R}_{g_1, \dots, g_l}(T)$ is a G_δ -subset of X . Without loss of generality, let μ be an arbitrarily given ergodic Borel probability measure of $G \curvearrowright_T X$ such that $\text{supp}(\mu)$ does not consist of a fixed point of T ; otherwise $\mu(\mathcal{R}_{g_1, \dots, g_l}(T)) = 1$. By contradiction, let $\mu(\mathcal{R}_{g_1, \dots, g_l}(T)) < 1$; then since $\mathcal{R}_{g_1, \dots, g_l}(T)$ is T -invariant, it follows at once that

$$Y = X - \mathcal{R}_{g_1, \dots, g_l}(T)$$

is an T -invariant Borel set of μ -measure 1. We now consider $G \curvearrowright_T (Y, \mu)$ instead of $G \curvearrowright_T X$. We shall prove that there is a (g_1, \dots, g_l) -recurrent point of $G \curvearrowright_T X$ that lies in Y . This is a contradiction.

Since Y is a Borel subset of X and μ is regular, by Theorem 0.3 it follows that there can be found a closed subset Y_0 of X with

$$Y_0 \subset Y, \quad 0 < \mu(Y_0) \leq \frac{1}{2} \quad \text{and} \quad \text{diam}(Y_0) \leq 1$$

and there is some element $\tau_1 > R_1$ in R (i.e. $\tau_1 \in R$, $\tau_1 \notin R_1$) so that

$$\mu(Y_0 \cap g_1^{-\tau_1} Y_0 \cap \dots \cap g_l^{-\tau_1} Y_0) > 0.$$

Then there can be found a closed subset Y_1 of X with

$$Y_1 \subset Y_0 \cap g_1^{-\tau_1} Y_0 \cap \cdots \cap g_l^{-\tau_1} Y_0, \quad 0 < \mu(Y_1) \leq \frac{1}{2^2}, \quad \text{diam}(Y_1) \leq \frac{1}{2},$$

Next by Theorem 0.3 again, there can be found an element $\tau_2 > R_2$ in R (i.e. $\tau_2 \in R, \tau_2 \notin R_2$) with

$$\mu(Y_1 \cap g_1^{-\tau_2} Y_1 \cap \cdots \cap g_l^{-\tau_2} Y_1) > 0.$$

Further there can be found a closed subset Y_2 of X with

$$Y_2 \subset Y_1 \cap g_1^{-\tau_2} Y_1 \cap \cdots \cap g_l^{-\tau_2} Y_1, \quad 0 < \mu(Y_2) \leq \frac{1}{2^3}, \quad \text{diam}(Y_2) \leq \frac{1}{2^2},$$

and an element $\tau_3 > R_3$ in R with

$$\mu(Y_2 \cap g_1^{-\tau_3} Y_2 \cap \cdots \cap g_l^{-\tau_3} Y_2) > 0.$$

Continuing this process without end and noting that $Y_n \subseteq Y_{n-1}$ for $n = 1, 2, \dots$ and, besides, that $\text{diam}(Y_n) \leq 1/2^n$, we obtain because of the compactness of the space X a point $q \in Y$ as the intersection of the sets Y_n :

$$\{q\} = \bigcap_{n=1}^{\infty} Y_n.$$

Since

$$g_i^{\tau_n} q \in Y_{n-1} \quad \text{for } n = 1, 2, \dots \text{ and } i = 1, \dots, l,$$

we can see that $\tau_n \rightarrow +\infty$ and

$$g_i^{\tau_n} q \rightarrow q \quad \text{as } n \rightarrow +\infty$$

simultaneously for $i = 1, \dots, l$. We thus arrive at a contradiction.

This completes the proof of Theorem 5.5. \square

Now combining Theorem 5.5 with Lemma 5.4 follows at once the following result:

Proposition 5.6. *Given any $g_1, \dots, g_l \in G$, the (g_1, \dots, g_l) -multiple recurrent points of $G \curvearrowright_T X$ form a residual and full probability set in the multiple Birkhoff center $\Omega_\gamma(T)$.*

Corollary 5.7. *If G is countable, then the multiply recurrent points of $G \curvearrowright_T X$ form a G_δ set of full probability, which is residual in $\Omega_\gamma(T)$.*

Proof. Since G is countable, hence $\bigcap_{l \in \mathbb{N}} \bigcap_{(g_1, \dots, g_l) \in G^l} \mathcal{R}_{g_1, \dots, g_l}(T)$ is G_δ in X and residual in $\Omega_\gamma(T)$ and of full probability. This proves the corollary. \square

Particularly under the discrete topology, \mathbb{Z}^d is a countable Noetherian \mathbb{Z} -module. Next we will consider a \mathbb{Z} -acting dynamical system. Let $T: X \rightarrow X$ be a homeomorphism of the compact metric space X , which has the multiple Birkhoff center $M_\gamma(T)$ and $([-n, \dots, n])_1^\infty$ is a summing sequence in $(\mathbb{Z}, +)$.

In 1994 [27], by using the topological “ergodic decomposition” theory developed in [17, 42] E. Glasner showed that if (X, T) is *minimal and topologically weakly mixing*, then for any $l \geq 1$ the $(1, 2, \dots, l)$ -recurrent points of T form a dense G_δ subset in X . Whenever (X, T) is *minimal*, then it is also well known that there always exists a dense G_δ set of points which are $(1, 2, \dots, l)$ -recurrent for any $l \geq 1$. This result appears scattered in many literature; see, e.g., [31,

Theorem 2.5]. Moreover, in [33, Theorem 3.12], D. Kwietniak et al. proved that the set of the $(1, 2, \dots, l)$ -recurrent points of T is of full probability.

The \mathbb{Z} -time version of Corollary 5.7 can be stated as follows:

Proposition 5.8. *Let $T: X \rightarrow X$ be a homeomorphism of the compact metric space X , which has the multiple Birkhoff center $M_\gamma(T)$. Then the set of all the multi-recurrent points of the single homeomorphism T is residual and of full probability in $M_\gamma(T)$.*

Here for the multiple recurrence, our sample times $t_1, \dots, t_l \in \mathbb{Z}$ are not necessarily to be positive.

We note that since there exists a dynamical system on the 2-dimensional torus \mathbb{T}^2 that is uniquely ergodic with the measure center $o = (0, 0)$ and has the multiple Birkhoff center \mathbb{T}^2 (cf. [36, Example 6.16]), it is complementary between the topological structure and the full probability of the set of all the multi-recurrent points. Propositions 5.6 and 5.8 show that for any topological dynamical system $G \curvearrowright_T X$, the multiply recurrent motions are by no means rare from both the topological viewpoint and the point of view of measure theory.

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